Formal methods, Fall 2008
Graph theory definitions
Assigned: 10/2, Due: 10/9

Most definitions here are from Tucker 1995.

(1) An **undirected graph** is a set of nodes together with a set of unordered edges. We represented this as a structure $G = \langle N, E\rangle$ where $E \subseteq \mathcal{P}(N)$ and every element of E has cardinality 2.

(2) A **directed graph** is a set of nodes together with a set of ordered edges. We represented this as a structure $G = \langle N, E\rangle$ where E is a set of pairs of elements of N. Note the E here is the same kind of object as a relation in the standard set-theoretic sense, and thus a directed graph is simply a particular way of construing a relation. (An undirected graph can be seen as a way of construing a symmetric relation.)

(3) A **tree** in the graph-theoretic sense can be defined in several ways. The definition we used most is for a directed, rooted tree: an ordered graph with a designated root node, such that there is a unique path from the root node to every other node in the tree.

(4) The **degree** of a node is the number of edges entering/leaving the node. The sum of the degrees of all vertices in a graph is equal to twice the number of edges.

(5) A **complete graph on n vertices** (written K_n) is a graph with n vertices in which every vertex is adjacent to every other.

(6) A graph is **planar** if it can be embedded into a 2-dimensional space without any crossing edges. (Many planar graphs can be drawn with crossing edges, but as long as there is some way of drawing them without, they are planar.) All trees are planar.

(7) A **subgraph** is a graph formed by a subset of vertices and edges of a larger graph.

(8) Two graphs are **isomorphic** if there is a one-to-one and onto function between the vertices of the first and the vertices of the second, such that a pair of vertices are adjacent in the first graph iff they are adjacent in the second. More precisely, two directed graphs G_1 and G_2 are isomorphic if there exists a bijection f from N_1 to N_2 such that xE_1y iff $f(x)E_2f(y)$, for any $x, y \in N_1$. Two undirected graphs G_1 and G_2 are isomorphic if there exists a bijection f from N_1 to N_2 such that $\{x, y\} \in E_1$ iff $\{f(x), f(y)\} \in E_2$. A specific bijection that has this property is referred to as a **graph isomorphism**.

If two graphs are isomorphic then corresponding subgraphs must be isomorphic. This leads to the main strategy for trying to find or disprove isomorphism – pick an isomorphic subgraph and try to build an isomorphism from there. If this fails for all isomorphic subgraphs of the same size, then you know that the graphs are not isomorphic.

Any instance of K_n is isomorphic to any other instance of K_n (for the same n).

(9) A graph **automorphism** is an isomorphism between a graph and itself. (For example, the identity function is always an automorphism.) A graph **homomorphism** is a function that has all the same properties as a graph isomorphism, except it need not be one-to-one.

(10) For the more restricted notion of tree we explored for linguistic purposes, see chapter 16 of the textbook. One fact not mentioned there is that the set of nodes together with the relation describing immediate domination describes the visual representation in terms of a tree in the sense defined in (3) above. The domination relation used in the textbook definition is the transitive closure of the immediate domination relation. The set of nodes together with D, or together with P, is a graph, but not a tree in the graph theory sense.

Bibliography

Tucker, Alan. 1995. *Applied combinatorics (third edition)*. John Wiley and Sons.